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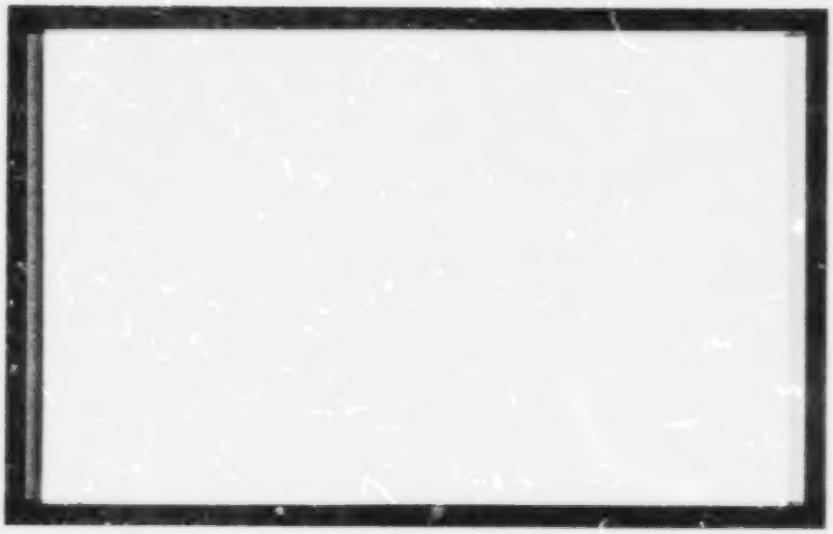

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MODELS FOR MULTIMODE MULTICOMMODITY LOCATION PROBLEMS WITH INTERDEPOT BALANCING REQUIREMENTS

by

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ABSTRACT

We present the problem of locating vehicle depots in an intercity freight transportation system with the objective of satisfying client demand for empty vehicles, while minimizing depot opening and operating costs, bidirectional client-depot transportation costs, as well as the costs of the interdepot movements necessary to counter the unbalancing of demand. We propose a class of models which has the form of a two-echelon multimode multicommodity location formulation with interdepot balancing requirements. We analyse the models and their properties to determine their underlying structure and characteristics to be used in subsequent algorithmic developments.

Key words : depot location, empty movement, empty balancing, multimode multicommodity freight transportation

RÉSUMÉ

Nous énonçons le problème de la localisation des dépôts de véhicules, pour le transport interurbain de marchandises, afin de minimiser les coûts associés à l'ouverture et au fonctionnement des dépôts, les coûts des mouvements bidirectionnels clients-dépôts, ainsi que ceux des mouvements interdépôts, nécessaires pour rééquilibrer le réseau. Le modèle que nous proposons se présente comme une formulation de localisation de dépôts multiniveaux, multimodes, multiproduits, avec exigences d'équilibrage des charges des dépôts. Nous présentons certaines variantes de modélisation, nous analysons les propriétés des modèles et dégageons les perspectives algorithmiques les plus intéressantes.

Mots-clefs : localisation de dépôts, mouvements à vide, équilibrage des charges de dépôt, transport multimode multiproduit des marchandises

1. INTRODUCTION

Our research was initially stimulated by the following industrial problem :

> A distribution or transportation firm delivers products to its clients using different types of vehicles of various transportation modes : railcars, trucks, containers, etc. Client demand being assumed known, the firm knows the exact number of loaded vehicles to ship to each client. After delivery and unloading by the clients, the empty vehicles are shipped back to a warehouse or depot. The same or other clients also need empty vehicles for subsequent loaded shipping with their own products. They order these empty vehicles from the depots and these demands are also supposed known by the transportation firm. After loading, the vehicles are shipped to their destination (normally one of the firm's network nodes). This shipment is carried out directly, or via the originating or another depot. The transportation system may use several modes (rail, road, combined mode, ...) and the transportation characteristics such as cost, travel time etc., are supposed known for each one of them. A major complication of the problem is due to the regional imbalance in empty vehicle availabilities and needs throughout the network : some areas lack vehicles of a given type, while others have too many of them. This requires the shipment of empty vehicles between depots for network balancing purposes. The global problem is therefore that of the location of the empty vehicle depots in order to satisfy client demand and to minimize the total costs of depot opening and operation and vehicle transportation from clients to depots, depots to clients and between depots.

This problem belongs to the general class of strategic problems of planning logistics and distribution networks. Yet, compared with the classical location-distribution problem, where movements are unidirectional from plants through warehouses to clients, the present problem owes its originality to the fact that empty freight vehicle movements are at the center of its preoccupations, which implies that back-and-forth traffic between depots and clients, as well as between depots, has to be explicitly considered. We have found no paper dealing specifically with this problem either in the empty vehicle management literature (Dejax and

Crainic, 1986), nor among papers dedicated to location-distribution problems (Krarup and Pruzan, 1983; Cornuejols, Nemhauser and Wolsey, 1983; Aikens 1985).

The functional form of the transportation costs determine the problem type : linear or not and, in the non-linear case, concave, convex or otherwise. Linear functions are normally used. Yet, concavity, for example, may facilitate the modeling of the economies of scale usually imbedded into transportation tariffs. The problem may also be either deterministic or stochastic, depending on hypotheses concerning the nature of demand, supply and transportation times.

We are currently working on the linear deterministic version of the problem. These hypotheses allow the separate planning of loaded and empty movements. We therefore concentrate our attention on the empty vehicle management problem, since it strongly interacts with the location of the depots.

Dejax et al. (1986) first treated this problem as part of a global study of the logistics system of a large European container transportation and distribution company. Their approach is to sequentially solve a classical depot location problem (using Erlenkotter's algorithm, 1978), followed by a minimum cost flow problem to compute the interdepot movements. Shortest path methods are then used to optimize the distribution of loaded containers.

Our research represents the continuation of their study and its general objective is to integrate the location and transportation problems. More specifically, this paper aims to present the various formulations of the problem and to analyze the models and their properties, in order to determine their underlying structure and characteristics which may be used in subsequent algorithmic developments.

In classical location theory, the class of problems which more

closely relates to our problem is usually called "the simple plant location problem" (Krarup and Pruzan, 1983) or "the uncapacitated facility location problem" (Cornuejols, Nemhauser and Wolsey, 1983). We will refer to this model using the notation of Krarup and Pruzan (1983) and will call it the (SPLP) problem.

2. NOTATION, HYPOTHESES AND DEFINITIONS

The current formulation considers empty movements only. We then assume, for each client, known supplies (equal, in fact, to the number of loaded vehicles received), and demands (equal to the number of loaded vehicles that need to be shipped) of vehicles of each type. We also assume that empty vehicles have to transit through a depot and that, accordingly, no client to client movements are allowed. In addition, since a strategic planning horizon is implicit in the problem, supply and demand are assumed nonsimultaneous. Consequently, empty vehicles are sent to the depot as soon as they are available and are not kept by the client to satisfy future needs.

Let

$C = \{i \mid \text{client}\}$

$n = |C|$

$D = \{j \mid \text{candidate depot}\}$

$m = |D|$

$P = \{p \mid \text{type of vehicle-commodity}\}$

$\ell = |P|$

$T = \{t \mid \text{transport mode}\}$.

These elements make up a conceptual network $G = (N,A)$, that generally parallels a physical network. Nodes are defined by clients and candidate depot sites : $N = C \cup D$. Arcs represent all possible movements, by all modes, between depots, from clients to depots and conversely :

$$A \subseteq (C \times D \times T) \cup (D \times D \times T) \cup (D \times C \times T)$$

As client to client movements are not allowed, arcs in the CxC set are not defined. Furthermore, for any forbidden movement between a client and a depot, the corresponding arc is assumed not included in A.

We now define

O_{ip}: supply at node i of vehicles of type p, $i \in C$, $p \in P$;

D_{ip}: demand at node i of vehicles of type p, $i \in C$, $p \in P$;

$y=(y_j)$: opening vector for depots

$$y_j = \begin{cases} 1 & \text{if depot i is open} \\ 0 & \text{otherwise} \end{cases} \qquad j \in D$$

$x=(x_{ijpt}, x_{jipt})$: vector of type p vehicle flows, on mode t, between client i and depot j, $i \in C$, $j \in D$, $p \in P$, $t \in T$;

$w=(w_{jkpt})$: vector of type p vehicle flows, on mode t, between depot j and depot k, $j,k \in D$, $p \in P$, $t \in T$;

$f_j(y,x,w)$: cost function to open and operate depot j, $j \in D$;

$c_{ijpt}(x)$ and $c_{jipt}(x)$: transportation cost functions of vehicles of type p, on mode t between client i and depot j, $i \in C$, $j \in D$, $p \in P$, $t \in T$;

$s_{jkpt}(w)$: transportation cost function of vehicles of type p, on mode t, between depots j and k, $j,k \in D$, $p \in P$, $t \in T$.

When linear costs and no capacities are assumed for the transportation links (as in the present formulation), traffic moves on the mode with the lowest unit cost. Consequently, we may simplify the notation by assuming that between each pair of nodes only one arc exists, in each direction, representing the mode (or mode combination) with the lowest unit transportation cost. The subscript t may then be omitted and we have :

$$\begin{aligned} c_{ijp}(x) &= c_{ijp}x_{ijp} && i \in C,\ j \in D,\ p \in P \\ c_{jip}(x) &= c_{jip}x_{jip} && j \in D,\ i \in C,\ p \in P \\ s_{jkp}(w) &= s_{jkp}w_{jkp} && j,k \in D,\ p \in P \\ f_j(y,x,w) &= f_j y_j && j \in D \ . \end{aligned}$$

We suppose that all unit costs are nonnegative. When this is not the case, simple transformations with no impact on the optimal solution may be used (Krarup et Pruzan, 1983) to ensure this.

It is interesting to note that the problem treated here may also be presented, in a more classical form, as a two-echelon location-distribution problem, from plants, through depots, to clients, with the requirement that depot workloads be equilibrated and all supply be shipped out of plants. Figure 1 illustrates this structure for the unimodal, single commodity case. Note that in our case, clients are present not only at the usual demand echelon but also at the, more unusual, supply (plants) echelon. This form of the problem indicates that there may be applications of the models developed here outside the area of empty vehicle management. In the present paper, however, we do not exploit further this multi-echelon form, since we prefer the more convenient identification of the network structure of the problem offered by the initial formulation.

3. THE LINEAR MODEL

The multimode multicommodity location-distribution model with interdepot balancing requirements may be written as :

(P1) Minimize $$z = \sum_{j \in D} f_j y_j$$

$$+ \sum_{p \in P} \left[\sum_{i \in C} \sum_{j \in D} (c_{ijp} x_{ijp} + c_{jip} x_{jip}) + \sum_{j \in D} \sum_{k \in D} s_{jkp} w_{jkp} \right]$$

subject to

$$\sum_{j \in D} x_{ijp} = O_{ip} \qquad \text{for all } i \in C,\ p \in P \quad (1)$$

$$\sum_{j \in D} x_{jip} = D_{ip}$$

$$x_{ijp} \leq M y_j \qquad \text{for all } i \in C,\ j \in D,\ p \in P \quad (2)$$
$$x_{jip} \leq M y_j$$

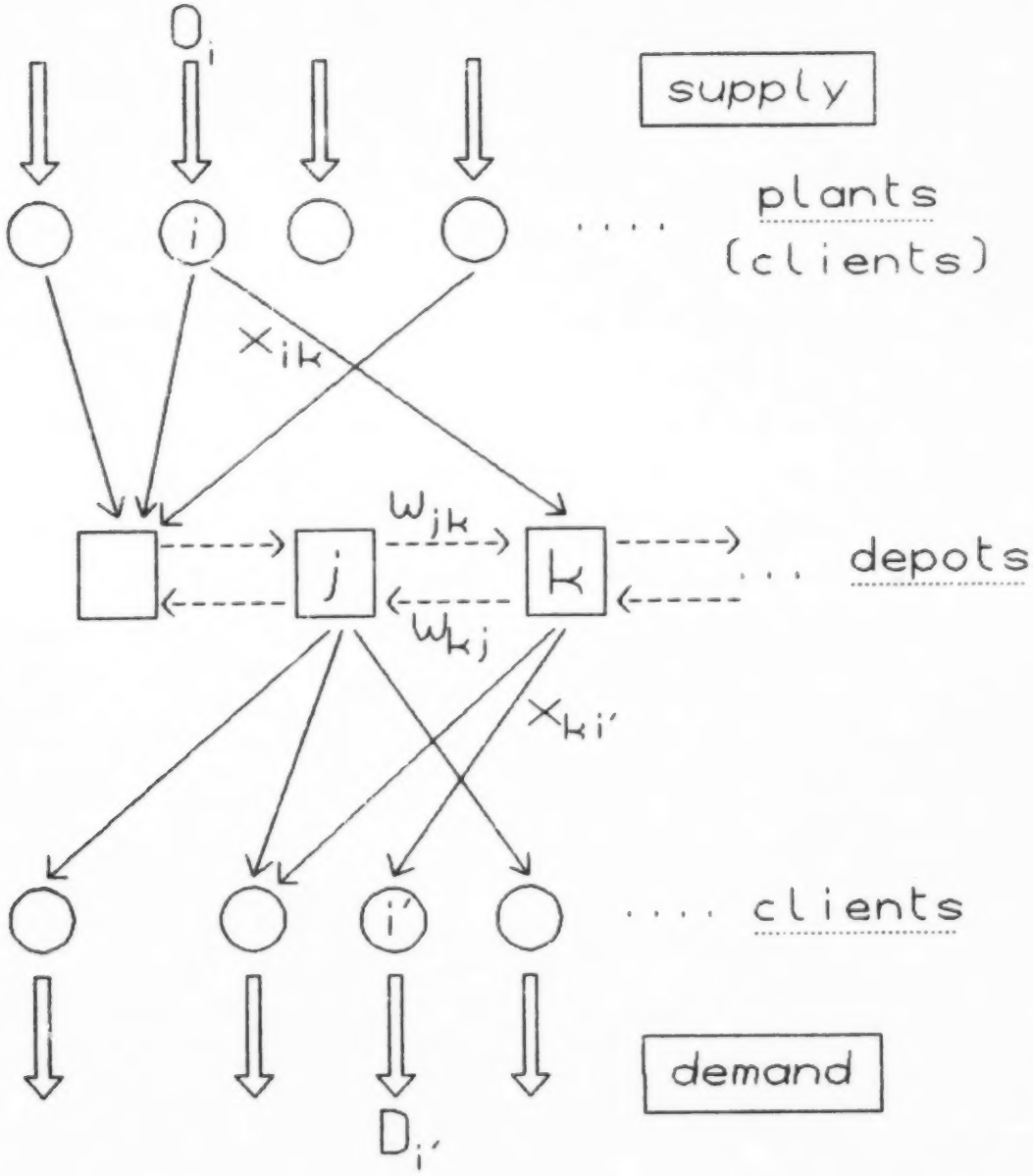

Figure 1 : Multi-echelon structure of the problem

$$\left.\begin{array}{l} w_{jkp} \leq My_j \\ w_{kjp} \leq My_j \end{array}\right\} \quad \text{for all } j,k \in D,\ p \in P \qquad (3)$$

$$\sum_{i \in C} x_{ijp} + \sum_{k \in D} w_{kjp} - \sum_{i \in C} x_{jip} - \sum_{k \in D} w_{jkp} = 0 \quad \text{for all } j \in D,\ p \in P \qquad (4)$$

$$x_{ijp} \geq 0 \qquad x_{jip} \geq 0 \qquad \text{for all } i \in C,\ j \in D,\ p \in P$$

$$w_{jkp} \geq 0 \qquad \text{for all } j,k \in D,\ p \in P$$

$$y_j \in \{0,1\} \qquad \text{for all } j \in D\ . \qquad (5)$$

Constraints (1) ensure that demand and supply requirements are met. Constraints (2) forbid exchanges between clients and closed depots (M is a suitably large constant), while constraints (3) do the same for interdepot movements. Finally, constraints (4) correspond to the interdepot balancing requirements.

Under somewhat mild hypotheses, the model may be considerably simplified. Supply and demand may thus advantageously replace the constant M in constraints (2), also tightening the feasible domain. Additionally, constraints (3) may be eliminated if interdepot transportation costs verify the triangle inequality. Formally :

Proposition 1 If the interdepot transportation cost functions verify:

$$s_{ikp} \leq s_{ijp} + s_{jkp} \qquad \text{for all } i,j,k \in D,\ p \in P \qquad (6)$$

then constraints (3) of model (P1)

$$\left.\begin{array}{l} w_{jkp} \leq My_j \\ w_{kjp} \leq My_j \end{array}\right\} \qquad \text{for all } j,k \in D,\ p \in P$$

are redundant.

Proof : In Annex.

Corollary When

$$s_{ikp} < s_{ijp} + s_{jkp} \qquad \text{for all } p \in P,\ i,j,k \in D$$

an optimal solution to the modified problem is feasible, thus optimal, for the initial model.

Note that the proof of Proposition 1 shows how to extract an optimal solution to the initial problem from the optimal solution of the modified model, when the interdepot transportation cost functions verify the triangle inequality with strict equality. We will then assume in the following that the strict inequality is verified for these functions.

The model may therefore be written as :

(L) Minimize $$z = \sum_{j \in D} f_j y_j + \sum_{p \in P} \left[\sum_{i \in C} \sum_{j \in D} (c_{ijp} x_{ijp} + c_{jip} x_{jip}) + \sum_{j \in D} \sum_{k \in D} s_{jkp} w_{jkp} \right]$$

subject to

$$\sum_{j \in D} x_{ijp} = O_{ip} \qquad \text{for all } i \in C,\ p \in P \quad (1)$$

$$\sum_{j \in D} x_{jip} = D_{ip}$$

$$x_{ijp} \leq O_{ip} y_j \qquad \text{for all } i \in C,\ j \in D,\ p \in P \quad (7)$$

$$x_{jip} \leq D_{ip} y_j$$

$$\sum_{i \in C} x_{ijp} + \sum_{k \in D} w_{kjp} - \sum_{i \in C} x_{jip} - \sum_{k \in D} w_{jkp} = 0 \qquad \text{for all } j \in D,\ p \in P \quad (4)$$

$$x_{ijp} \geq 0 \qquad x_{jip} \geq 0 \qquad \text{for all } i \in C,\ j \in D,\ p \in P$$

$$w_{jkp} \geq 0 \qquad \text{for all } j,k \in D,\ p \in P$$

$$y_j \in \{0,1\} \qquad \text{for all } j \in D \quad (5)$$

The "normalized" model offers an equivalent formulation, useful for duality purposes :

$$(LN)\quad \text{Minimize}\quad z = \sum_{j\in D} f_j y_j + \sum_{p\in P}\Big[\sum_{i\in C}\sum_{j\in D}(c_{ijp}O_{ip}x_{ijp} + c_{jip}D_{ip}x_{jip}) + \sum_{j\in D}\sum_{k\in D} s_{jkp}w_{jkp}\Big]$$

subject to

$$\sum_{j\in D} x_{ijp} = 1 \qquad \text{for all } i\in C,\ p\in P$$

$$\sum_{j\in D} x_{jip} = 1$$

$$x_{ijp} \leq y_j \qquad \text{for all } i\in C,\ j\in D,\ p\in P$$
$$x_{jip} \leq y_j$$

$$\sum_{i\in C} O_{ip}x_{ijp} + \sum_{k\in D} w_{kjp} - \sum_{i\in C} D_{ip}x_{jip} - \sum_{k\in D} w_{jkp} = 0 \qquad \text{for all } j\in D,\ p\in P$$

$$x_{ijp} \geq 0,\ x_{jip} \geq 0 \qquad \text{for all } i\in C,\ j\in D,\ p\in P$$

$$w_{jkp} \geq 0 \qquad \text{for all } j,k\in D,\ p\in P$$

$$y_j \in \{0,1\} \qquad \text{for all } j\in D$$

The triangle inequality hypothesis is assumed, in Proposition 1, for cost functions of the same class : interdepot transportation, on generally long distances, of large flows "consolidated" on transportation modes which are efficient for such movements. It is a realistic hypothesis, almost always verified in practice. This is however not the case when all cost functions are jointly considered. It is thus often less costly to use an indirect itinerary (e.g. client-intermediary depot-final depot), because the interdepot segment is more efficient that the direct, client-final depot movement. Yet, when the same hypothesis is verified for all transportation cost functions (as when, for example, all transportation costs are equally proportional to the distance traveled), the model may be further simplified, following this proposition :

Proposition 2 When for model (L), all cost functions satisfy the triangle inequality, particularly when $c_{ijp} + s_{jkp} > c_{ikp}$ for all $i \in C$, $j,k \in D$, $p \in P$, there is an optimal solution for (L) with $w_{jkp} = 0$ for all $j,k \in D$, $p \in P$.

The model may then be written as :

(LS) Minimize $$z = \sum_{j \in D} f_j y_j + \sum_{p \in P} \sum_{i \in C} \sum_{j \in D} (c_{ijp} x_{ijp} + c_{jip} x_{jip})$$

subject to

$$\sum_{j \in D} x_{ijp} = O_{ip} \qquad \text{for all } i \in C,\ p \in P$$

$$\sum_{j \in D} x_{jip} = D_{ip}$$

$$x_{ijp} \leq O_{ip} y_j \qquad \text{for all } i \in C,\ j \in D,\ p \in P$$

$$x_{jip} \leq D_{ip} y_j$$

$$\sum_{i \in C} x_{ijp} - \sum_{i \in C} x_{jip} = 0 \qquad \text{for all } j \in D,\ p \in P$$

$$x_{ijp} \geq 0,\ x_{jip} \geq 0 \qquad \text{for all } i \in C,\ j \in D,\ p \in P$$

$$y_j \in \{0,1\} \qquad \text{for all } j \in D.$$

This model has no practical interest except to emphasize a major distinction between our problem and the classical location model. It is well known (see for example Krarup and Pruzan, 1983) that the simple location problem (SPLP) has the single assignment property : all of a client's demand is satisfied from the open depot with the lowest transportation cost. For our problem, however, as depot supplies are exclusively generated by client shipments (via another depot, eventually), this property is not generally verified.

4. SUB-PROBLEMS AND RELAXATIONS

We now present formulations derived from the initial model (L), either by fixing the value of some variables or by relaxing constraints. These subproblems will be useful for the development of solution procedures for model (L).

When, in formulation (L), the y_j variables are set to $\bar{y}_j$, the problem becomes :

$$(L(\bar{y})) \quad \text{Minimize} \quad z = \sum_{p\in P} \Big[\sum_{i\in C} \sum_{j\in D} (c_{ijp}x_{ijp} + c_{jip}x_{jip}) + \sum_{k,j\in \bar{D}} s_{jkp}w_{jkp} \Big]$$

subject to

$$\sum_{j\in D} x_{ijp} = O_{ip} \qquad \text{for all } i\in C,\ p\in P$$

$$\sum_{j\in D} x_{jip} = D_{ip}$$

$$\sum_{i\in C} x_{ijp} + \sum_{k\in D} w_{kjp} - \sum_{i\in C} x_{jip} - \sum_{k\in D} w_{jkp} = 0 \qquad \text{for all } j\in \bar{D},\ p\in P$$

$$x_{ijp}, x_{jip}, w_{jkp} \geq 0 \qquad \text{for all } i\in C,\ p\in P,\ j,k\in D$$

$$x_{ijp} = x_{jip} = 0 \qquad \text{for all } j\in D\setminus\bar{D},\ i\in C,\ p\in P$$

$$w_{jkp} = w_{kjp} = 0 \qquad \text{for all } j\in D\setminus\bar{D},\ k\in D,\ p\in P$$

$$\bar{D} = \{j \mid \bar{y}_j = 1\} .$$

This problem has the following property :

Proposition 3 $(L(\bar{y}))$ is an uncapacitated minimum cost multiflow problem.

Proof : Consider the network $G=(N,A)$ displayed in Figure 1 and defined as follows :

$N = C \cup \bar{D} \cup C'$ and $A = A_1 \cup A_2 \cup A_3$ where :

$C' = \{r \mid r=n+i, \text{ for all } i\in C\}$

and

$$A_1 = \{(r,s) \mid r \in C,\ s \in \bar{D}\}$$

$$A_2 = \{(r,s) \mid r \in \bar{D},\ s \in C'\}$$

$$A_3 = \{(r,s) \mid r,s \in \bar{D}\}$$

In addition, let :

$$\begin{cases} a_{ijp} = c_{ijp} \text{ and } z_{ijp} = x_{ijp} & \text{for all } (i,j) \in A_1, \\ a_{ji'p} = c_{jip} \text{ and } z_{ji'p} = x_{jip} & \text{for all } (j,i') \in A_2 \\ & \text{where } i' = n+i \text{ and } i \in C, \\ a_{jkp} = s_{jkp} \text{ and } z_{jkp} = w_{jkp} & \text{for all } (j,k) \in A_3, \end{cases}$$

for all $p \in P$.

Problem $(L(\bar{y}))$ may then be written as follows :

$$(L(\bar{y})) \quad \text{Minimize } z = \sum_{p \in P} \sum_{(r,s) \in A} a_{rsp} z_{rsp}$$

subject to

$$\sum_{(r,s) \in A} z_{rsp} - \sum_{(s,r) \in A} z_{srp} = \begin{cases} 0_{rp} & \text{for all } r \in C,\ p \in P \\ 0 & \text{for all } r \in \bar{D},\ p \in P \\ -D_{rp} & \text{for all } r \in C',\ p \in P \end{cases}$$

$$z_{rsp} \geq 0 \qquad \text{for all } (r,s) \in A,\ p \in P$$

which is a minimum cost multiflow problem. □

Note that $(L(\bar{y}))$ can be decomposed into ℓ mutually exclusive minimum cost flow subproblems, one for each commodity.

It is frequently necessary to compute a lower bound on the optimal solution of problem (L) by relaxing constraints (5). We define then the strong and weak linear relaxations of (L), denoted by (SRL) and (WRL) respectively, as the following problems :

$$\text{Minimize } z = \sum_{j\in D} f_j y_j + \sum_{p\in P} \left[\sum_{i\in C} \sum_{j\in D} (c_{ijp}x_{ijp} + c_{jip}x_{jip}) + \sum_{j\in D} \sum_{k\in D} s_{kjp}w_{kjp} \right]$$

subject to

$$\sum_{j\in D} x_{ijp} = O_{ip} \qquad \text{for all } i\in C,\ p\in P$$

$$\sum_{j\in D} x_{jip} = D_{ip}$$

$$x_{ijp} \leq O_{ip} y_j$$

(SRL) $$x_{jip} \leq D_{ip} y_j \qquad \text{for all } i\in C,\ j\in D,\ p\in P \quad (8)$$

(WRL) $$\sum_{p\in P} \sum_{i\in C} (x_{ijp} + x_{jip}) \leq \left[\sum_{p\in P} \sum_{i\in C} (O_{ip} + D_{ip}) \right] y_j \qquad \text{for all } j\in D \quad (9)$$

$$\sum_{i\in C} x_{ijp} + \sum_{k\in D} w_{kjp} - \sum_{i\in C} x_{jip} - \sum_{k\in D} w_{jkp} = 0 \qquad \text{for all } j\in D,\ p\in P$$

$$x_{ijp}, x_{jip} \geq 0 \qquad \text{for all } i\in C,\ j\in D,\ p\in P$$

$$w_{kjp} \geq 0 \qquad \text{for all } k, j\in D,\ p\in P$$

$$y_j \geq 0 \qquad \text{for all } j\in D.$$

The equivalent formulations for the normalized model are called (SRLN) and (WRLN) respectively.

Problems (SRL) and (L($\bar{y}$)) are almost similar, except for constraints (8), which do not appear in the latter. These constraints may be seen as variable arc capacities on the x_{ijp} and x_{jip} flows. A problem such as (SRL) is usually more difficult to solve than standard minimum cost flow problems (see Schrage, 1975, 1978). This is not, however, the case for (WRL); simply note that at the optimum we always have :

$$y_j = \sum_{p\in P} \sum_{i\in C} (x_{ijp} + x_{jip}) / \Delta \qquad \text{for all } j\in D \quad (10)$$

where $$\Delta = \sum_{p\in P} \sum_{i\in C} (O_{ip} + D_{ip})$$

This result is due to the fact that y_j appears in only one type (9) constraint in the problem and that its cost, f_j, is nonnegative. Thus, we can rewrite (WRL) by substituting the expression defined in (10) to y_j, to obtain an equivalent problem (WRL1) :

$$\text{(WRL1)} \quad \text{Minimize} \quad z = \sum_{p\in P}\Big[\sum_{i\in C}\sum_{j\in D}\big((c_{ijp}+f_j/\Delta)x_{ijp}+(c_{jip}+f_j/\Delta)x_{jip}\big) + \sum_{k\in D}\sum_{j\in D} s_{kjp}w_{kjp}\Big]$$

subject to

$$\sum_{j\in D} x_{ijp} = O_{ip} \qquad \text{for all } i\in C,\ p\in P$$

$$\sum_{j\in D} x_{jip} = D_{ip}$$

$$\sum_{i\in C} x_{ijp} + \sum_{k\in D} w_{kjp} - \sum_{i\in C} x_{jip} - \sum_{k\in D} w_{jkp} = 0 \qquad \text{for all } j\in D,\ p\in P$$

$$x_{ijp}, x_{jip}, w_{kjp} \geq 0 \qquad \text{for all } i\in C,\ k,j\in D,\ p\in P\ .$$

This problem has the following propriety :

Proposition 4 (WRL1) is an uncapacitated minimum cost multiflow problem.

Proof : Simply note that (WRL1) and $(L(\bar{y}))$ have similar forms. The proof is then analogous to the one developed for proposition 3.

(WRL1) can also be decomposed, by commodity, into ℓ mutually exclusive minimum cost flow subproblems.

Although (WRL) is easier to solve than (SRL), it generates a worse lower bound on the optimal solution of problem (L). As a matter of fact (WRL) is itself a relaxation of (SRL) : its ℓ constraints of type (9) are linear combinations of the ($2 \times n \times m \times \ell$) constraints (8) of problem (SRL).

An upper bound on the optimal solution of (L) may be obtained by defining a feasible solution (y^*, x^*, w^*) for (L) as follows :

$$y_j^* = \begin{cases} 1 & \text{if there exists at least } i \in C \text{ and } p \in P, \text{ such that } x^*_{ijp} > 0 \text{ or } x^*_{jip} > 0 \\ 0 & \text{otherwise} \end{cases}$$

for all $j \in D$, where (x^*, w^*) is an optimal solution for problem (WRL1). Note that this bound may not be very good.

Constraints (7) in problem (L) are "complicating". To dispose of them, two approaches may be used : the penalty method (Zangwill, 1967) and the Lagrangian relaxation of the problem (Bertsekas, 1982, chap. 4). The first approach gives rise to the following formulation, denoted by (PRL) :

$$\text{(PRL) Minimize } z = \sum_{j \in D} f_j y_j + \sum_{p \in P} \left[\sum_{i \in C} \sum_{j \in D} (c_{ijp} x_{ijp} + c_{jip} x_{jip}) + \sum_{k,j \in D} s_{kjp} w_{kjp} \right]$$

$$+ \sum_{p \in P} \sum_{i \in C} \sum_{j \in D} \left[K_{ijp} (\text{Min}\{0, O_{ip} y_j - x_{ijp}\})^2 \right.$$

$$\left. + K_{jip} (\text{Min}\{0, D_{ip} y_j - x_{jip}\})^2 \right]$$

subject to

$$\sum_{j \in D} x_{ijp} = O_{ip} \qquad \text{for all } i \in C,\ p \in P$$

$$\sum_{j \in D} x_{jip} = D_{ip}$$

$$\sum_{i \in C} x_{ijp} + \sum_{k \in D} w_{kjp} - \sum_{i \in C} x_{jip} - \sum_{k \in D} w_{jkp} = 0 \qquad \text{for all } j \in D,\ p \in P$$

$$x_{ijp}, x_{jip}, w_{kjp} \geq 0 \qquad \text{for all } i \in C,\ j,k \in D,\ p \in P$$

$$y_j \in \{0,1\} \qquad \text{for all } j \in D$$

where K_{ijp} and K_{jip} are strictly positive penalty constants. This problem has the following propriety when dropping the zero-one constraint on the y variables :

Proposition 5 The strong relaxation of (PRL) is an uncapacitated minimum cost multiflow problem with convex arc costs.

Proof : Note first that the constraint sets of the relaxed problem and of problem (L($\bar{y}$)) are similar. Thus, by proposition 3, (PRL) has a minimum cost multiflow network structure. In addition, the last part of the objective function, related to the penalty functions, is clearly convex; the objective function of (PRL) is then convex, which completes the proof. □

In order to solve (PRL), it might be worthwhile to adapt the relaxation techniques developed by Bertsekas (1985) (see also Bertsekas, Hossein and Tseng, 1985) for minimum cost network problems. We are currently exploring this avenue.

The Lagrangian relaxation of problem (L), with respect to constraints (7), implies solving the following problem, where γ_{ijp} and γ_{jip} are given nonnegative Lagrange multipliers :

$$\begin{aligned}(\text{LRL}(\gamma))\ \text{Minimize}\quad z = & \sum_{j\in D} f_j y_j + \sum_{p\in P}\Big[\sum_{i\in C}\sum_{j\in D}(c_{ijp}x_{ijp}+c_{jip}x_{jip}) \\ & + \sum_{j\in D}\sum_{k\in D} s_{kjp}w_{kjp}\Big] \\ & + \sum_{p\in P}\sum_{i\in C}\sum_{j\in D}\big[\gamma_{ijp}(x_{ijp}-O_{ip}y_j) \\ & + \gamma_{jip}(x_{jip}-D_{ip}y_j)\big]\end{aligned}$$

subject to

$$\sum_{j\in D} x_{ijp} = O_{ip} \qquad \text{for all } i\in C,\ p\in P$$

$$\sum_{j\in D} x_{jip} = D_{ip} \qquad \text{for all } i\in C,\ p\in P$$

$$\sum_{i\in C} x_{ijp} + \sum_{k\in D} w_{kjp} - \sum_{i\in C} x_{jip} - \sum_{k\in D} w_{jkp} = 0 \qquad \text{for all } j\in D,\ p\in P$$

$$x_{ijp}, x_{jip}, w_{kjp} \geq 0 \qquad \text{for all } i\in C,\ j,k\in D,\ p\in P$$

$$y_j \in \{0,1\} \qquad \text{for all } j\in D$$

This problem also has a multiflow network structure :

Proposition 6 The strong relaxation of (LRL(γ)) is a linear uncapacitated minimum cost multiflow problem.

Proof : Note that the constraint sets of the relaxed problem and of problem (L($\bar{y}$)) are similar. Moreover, the objective function of (LRL(γ)) is clearly linear. Thus, by proposition 3, the problem has a linear minimum cost multiflow network structure, which completes the proof. □

In order to find an optimal solution for problem (L), one may then first solve (LRL(γ)) for a given γ and then adjust the multipliers to decrease the value of the objective, while satisfying the following complementary conditions :

$$\gamma_{ijp}(x_{ijp} - O_{ip}y_j) = 0$$
$$\gamma_{jip}(x_{jip} - D_{ip}y_j) = 0 \qquad \text{for all } i \in C,\ j \in D,\ p \in P.$$

This procedure must be repeated until convergence is obtained. We are planning to test this approach in the near future.

5. DUALIZATION

To characterize the dual properties of the initial problem (L), let us consider the strong linear relaxation of the normalized problem :

$$\text{(SRLN)} \quad \text{Minimize } z^P(y,x,w) = \sum_{j \in D} f_j y_j + \sum_{p \in P} \Big[\sum_{i \in C} \sum_{j \in D} (C_{ijp}x_{ijp} + C_{jip}x_{jip}) + \sum_{k \in D} \sum_{j \in D} s_{kjp}w_{kjp} \Big]$$

subject t

$$\sum_{j\in D} x_{ijp} = 1 \qquad \text{for all } i\in C,\ p\in P \qquad (u_{ip})$$

$$\sum_{j\in D} x_{jip} = 1 \qquad \text{for all } i\in C,\ p\in P \qquad (v_{ip})$$

$$x_{ijp} \le y_j \qquad \text{for all } i\in C,\ j\in D,\ p\in P \qquad (\gamma_{ijp})$$

$$x_{jip} \le y_j \qquad \text{for all } i\in C,\ j\in D,\ p\in P \qquad (\gamma_{jip})$$

$$\sum_{i\in C} O_{ip}x_{ijp} + \sum_{k\in D} w_{kjp} - \sum_{i\in C} D_{ip}x_{jip} - \sum_{k\in D} w_{jkp} = 0 \qquad (\lambda_{jp})$$

$$\text{for all } j\in D,\ p\in P$$

$$x_{ijp}, x_{jip}, w_{kjp} \ge 0 \qquad \text{for all } i\in C,\ j,k\in D,\ p\in P$$

$$y_j \ge 0 \qquad \text{for all } j\in D$$

where :

$$C_{ijp} = O_{ip}c_{ijp}$$

$$C_{jip} = D_{ip}c_{jip} \qquad \text{for all } i\in C,\ j\in D,\ p\in P\ . \quad (11)$$

Let (y^*, x^*, w^*) be an optimal solution for this problem. In addition, suppose that u_{ip}, v_{ip}, γ_{ijp}, γ_{jip} and λ_{jp} are dual variables related to the constraints as defined above. The dual formulation of (SRLN) is then as follows :

(DLN) Maximize $z^D(u,v,\gamma,\lambda) = \sum_{p\in P}\sum_{i\in C}(u_{ip}+v_{ip})$

subject to

$$u_{ip} - \gamma_{ijp} + O_{ip}\lambda_{jp} \le C_{ijp}$$

$$v_{ip} - \gamma_{jip} - D_{ip}\lambda_{jp} \le C_{jip} \qquad \text{for all } i\in C,\ j\in D,\ p\in P$$

$$\sum_{p\in P}\sum_{i\in C}(\gamma_{ijp}+\gamma_{jip}) \le f_j \qquad \text{for all } j\in D$$

$$\lambda_{jp} - \lambda_{kp} \le s_{kjp} \qquad \text{for all } k,j\in D,\ p\in P$$

$$\gamma_{ijp},\ \gamma_{jip} \ge 0 \qquad \text{for all } i\in C,\ j\in D,\ p\in P.$$

This problem has a very special structure when the λ_{jp} variables are set to values ($\bar{\lambda}_{jp}$) that verify the following inequality system :

$$\bar{\lambda}_{jp} - \bar{\lambda}_{kp} \leq s_{kjp} \qquad \text{for all } k,j \in D,\ p \in P.$$

The following formulation is then obtained :

$$(DLN(\bar{\lambda}))\quad \text{Maximize}\quad z^D(u,v,\gamma|\bar{\lambda}) = \sum_{p\in P}\sum_{i\in C}(u_{ip}+v_{ip})$$

subject to

$$u_{ip} - \gamma_{ijp} \leq (C_{ijp}-O_{ip}\bar{\lambda}_{jp}) \qquad \text{for all } i \in C,\ j \in D,\ p \in P$$
$$v_{ip} - \gamma_{jip} \leq (C_{jip}+D_{ip}\bar{\lambda}_{jp})$$

$$\sum_{p\in P}\sum_{i\in C}(\gamma_{ijp}+\gamma_{jip}) \leq f_j \qquad \text{for all } j \in D$$

$$\gamma_{ijp},\gamma_{jip} \geq 0 \qquad \text{for all } i \in C,\ j \in D,\ p \in P.$$

The dual problem of $(DLN(\bar{\lambda}))$ is then as follows :

$$(SRLN(\bar{\lambda}))\quad \text{Minimize}\quad z^P(y,x|\bar{\lambda}) = \sum_{j\in D} f_j y_j + \sum_{p\in P}\sum_{i\in C}\sum_{j\in D}\{(C_{ijp}-O_{ip}\bar{\lambda}_{jp})x_{ijp}+(C_{jip}+D_{ip}\bar{\lambda}_{jp})x_{jip}\}$$

subject to

$$\sum_{j\in D} x_{ijp} = 1 \qquad \text{for all } i \in C,\ p \in P$$

$$\sum_{j\in D} x_{jip} = 1$$

$$x_{ijp} \leq y_j \qquad \text{for all } i \in C,\ j \in D,\ p \in P$$

$$x_{jip} \leq y_j$$

$$y_j \geq 0 \qquad \text{for all } j \in D$$

$$x_{ijp},x_{jip} \geq 0 \qquad \text{for all } i \in C,\ j \in D,\ p \in P.$$

This problem clearly is the strong linear relaxation of an uncapacitated two-echelon plant location model, displaying the network structure

illustrated in figure 1, but without interdepot arcs. It can then be solved by the dual method proposed by Erlenkotter (1978).

The $\bar{\lambda}_{jp}$ constants characterize the desequilibrium state of the system at the depots; in particular :

i) If $\bar{\lambda}_{jp}$ is a relatively large positive number, depot j should attract more flow of commodity p than it can send to clients, because the initial transportation cost associated with x_{ijp}, C_{ijp}, decreases while the cost of x_{jip} increases. $\bar{\lambda}_{jp}$ seems then to indicate that depot j tends to have a shortage of commodity p.

ii) On the other hand, if $\bar{\lambda}_{jp}$ is a relatively small negative number, we observe the opposite phenomenon : depot j should send more flow of commodity p to clients than it can receive from them. In this case, $\bar{\lambda}_{jp}$ seems to indicate that depot j tends to have an excess of commodity p.

Let $(\bar{y},\bar{x})$ be an optimal solution for problem (SRLN($\bar{\lambda}$)). When setting the y and x variables of problem (SRLN) to these values, we obtain the following problem :

(LN($\bar{y},\bar{x}$)) Minimize $z^P(w\ \bar{y},\bar{x}) = \sum_{p\in P} \sum_{k\in D} \sum_{j\in D} s_{kjp} w_{kjp}$

subject to

$$\sum_{k\in D} w_{kjp} - \sum_{k\in D} w_{jkp} = \sum_{i\in C} (D_{ip}\bar{x}_{jip} - O_{ip}\bar{x}_{ijp}) \qquad \text{for all } j\in D,\ p\in P$$

$$w_{kjp} \geq 0 \qquad \text{for all } k,j\in D, p\in P.$$

(LN($\bar{y},\bar{x}$)), as (L($\bar{y}$)) and (WRL1), can be decomposed into ℓ mutually exclusive minimum cost flow subproblems, one for each commodity . Let $\bar{w}$ be an optimal solution for this problem. We have enough information to compute bounds on the optimal solution of (SRLN), $z^P(y^*,x^*,w^*)$:

Proposition 7

$$z^P(\bar{y},\bar{x}|\ \bar{\lambda}) \leq z^P(y^*,x^*,w^*) \leq z^P(\bar{y},\bar{x},\bar{w})$$

Proof

i) First, recall that $z^P(\bar{y},\bar{x}|\bar{\lambda})$ is the optimal value of the objective function of (SRLN($\bar{\lambda}$)); thus it is also equal to the optimal value of the objective function of its dual, (DLN($\bar{\lambda}$)). It follows that $z^P(\bar{y},\bar{x}|\bar{\lambda})$ is a lower bound on the optimal solution of (DLN), and hence on $z^P(y^*,x^*,w^*)$.

ii) On the other hand, $(\bar{y},\bar{x},\bar{w})$ is clearly a feasible solution for problem (SRLN); consequently, $z^P(\bar{y},\bar{x},\bar{w})$ is an upper bound on the optimal solution of (SRLN), i.e. on $z^P(y^*,x^*,w^*)$. □

Note, finally, that solving first (SRLN($\bar{\lambda}$)) and then (SRLN($\bar{y},\bar{x}$)), where $\bar{\lambda}_{jp} = 0$ for all $j\in D$ and $p\in P$, corresponds in fact to the heuristic proposed by Dejax et al (1986) for model (L).

Another interesting result, related to the dual formulation (DLN), occurs when the γ_{ijp} et γ_{jip} variables are fixed to $\bar{\bar{\gamma}}_{ijp}$ and $\bar{\bar{\gamma}}_{jip}$ respectively. These values must be such that :

$$\sum_{p\in P} (\bar{\bar{\gamma}}_{ijp}+\bar{\bar{\gamma}}_{jip}) \leq f_j \qquad \text{for all } j\in D$$

where $\bar{\bar{\gamma}}_{jip} \geq 0$ and $\bar{\bar{\gamma}}_{jip} \geq 0$ for all $i\in C$, $j\in D$, $p\in P$.

We then obtain the following problem :

$$(DLN(\bar{\bar{\gamma}})) \quad \text{Maximize} \quad z^D(u,v,\lambda|\bar{\bar{\gamma}}) = \sum_{p\in P} \sum_{i\in C} (u_{ip}+v_{ip})$$

subject to

$$u_{ip} + O_{ip}\lambda_{jp} \leq (C_{ijp}+\bar{\bar{\gamma}}_{ijp})$$
$$v_{ip} - D_{ip}\lambda_{jp} \leq (C_{jip}+\bar{\bar{\gamma}}_{jip}) \qquad \text{for all } i\in C,\ j\in D,\ p\in P$$

$$\lambda_{jp}-\lambda_{kp} \leq s_{kjp} \qquad \text{for all } k,j\in D,\ p\in P.$$

The dual formulation of this problem is then as follows :

$$(SRLN(\bar{\bar{\gamma}}))\quad \text{Minimize}\quad z^P(x,w|\bar{\bar{\gamma}}) = \sum_{p\in P}\Big[\sum_{k\in D}\sum_{j\in D} s_{kjp}w_{kjp}$$

$$+ \sum_{i\in C}\sum_{j\in D}((C_{ijp}+\bar{\bar{\gamma}}_{ijp})x_{ijp}+(C_{jip}+\bar{\bar{\gamma}}_{jip})x_{jip})\Big]$$

subject to

$$\sum_{j\in D} x_{ijp} = 1 \qquad \text{for all } i\in C,\ p\in P$$

$$\sum_{j\in D} x_{jip} = 1 \qquad \text{for all } i\in C,\ p\in P$$

$$\sum_{i\in C} O_{ip}x_{ijp} + \sum_{k\in D} w_{kjp} - \sum_{i\in C} D_{ip}x_{jip} - \sum_{k\in D} w_{jkp} = 0$$

$$\text{for all } j\in D,\ p\in P$$

$$x_{ijp},\ x_{jip},\ w_{kjp} \geq 0 \qquad \text{for all } i\in C,\ j,k\in D,\ p\in P.$$

$(SRLN(\bar{\bar{\gamma}}))$ also has a minimum cost multiflow network structure, as noted in the following proposition :

Proposition 8 $(SRLN(\bar{\bar{\gamma}}))$ is an uncapacitated minimum cost multiflow problem.

Proof Problems $(SRLN(\bar{\bar{\gamma}}))$ and $(L(\bar{y}))$ having similar forms, the proof is analogous to the one developed for proposition 3. □

The $\bar{\bar{\gamma}}_{ijp}$ and $\bar{\bar{\gamma}}_{jip}$ constants adjust the client-depot and depot-client transportation costs to take into account the operating and opening costs of depot j. To see this, consider the complementary conditions between constraints (8) and the γ variables. We have :

$$\gamma_{ijp} > 0 \Rightarrow y_j = x_{ijp}$$

$$\gamma_{jip} > 0 \Rightarrow y_j = x_{jip}$$

and for all $i\in C,\ j\in D,\ p\in P$

$$y_j > x_{ijp} \Rightarrow \gamma_{ijp} = 0$$

$$y_j > x_{jip} \Rightarrow \gamma_{jip} = 0$$

$\bar{\bar{\gamma}}_{ijp}$ and $\bar{\bar{\gamma}}_{jip}$ can then be seen as prices to be paid in order to increase the value of y_j, i.e. to "open" depot j.

Let $(\bar{\bar{x}},\bar{\bar{w}})$ be an optimal solution for (SRLN($\bar{\bar{\gamma}}$)), and let $\bar{\bar{y}}$ be defined as follows :

$$\bar{\bar{y}}_j = \max_{i\in C, p\in P} \{\bar{\bar{x}}_{ijp}, \bar{\bar{x}}_{jip}\} \qquad \text{for all } j\in D.$$

Note that $\bar{\bar{y}}_j$ will always be less or equal to 1, because $\bar{\bar{x}}_{ijp}$ and $\bar{\bar{x}}_{jip}$ are such that :

$$\sum_{j\in D} \bar{\bar{x}}_{ijp} = 1 \quad \text{and} \quad \sum_{j\in D} \bar{\bar{x}}_{jip} = 1 \qquad \text{for all } i\in C \text{ and } p\in P.$$

We can again compute bounds on the optimal solution of (SRLN), $z^P(y^*,x^*,w^*)$ using the above information :

Proposition 9

$$z^P(\bar{\bar{x}},\bar{\bar{w}}|\bar{\bar{\gamma}}) \leq z^P(y^*,x^*,w^*) \leq z^P(\bar{\bar{y}},\bar{\bar{x}},\bar{\bar{w}}) \quad .$$

Proof

i) First, recall that $z^P(\bar{\bar{x}},\bar{\bar{w}}|\bar{\bar{\gamma}})$ is the optimal value of the objective function of (SRLN($\bar{\bar{\gamma}}$)); thus it is also equal to the optimal value of the objective function of its dual (DLN($\bar{\bar{\gamma}}$)). It follows that $z^P(\bar{\bar{x}},\bar{\bar{w}}|\bar{\bar{\gamma}})$ is a lower bound on the optimal solution of (DLN), and hence on $z^P(y^*,x^*,w^*)$.

ii) On the other hand, $(\bar{\bar{y}},\bar{\bar{x}},\bar{\bar{w}})$ is clearly a feasible solution for problem (SRLN); consequently, $z^P(\bar{\bar{y}},\bar{\bar{x}},\bar{\bar{w}})$ is an upper bound on the optimal solution of (SRLN), i.e. $z^P(y^*,x^*,w^*)$. □

Let us close this section by mentioning that problems (SRLN($\bar{\bar{\lambda}}$)) and (SRLN($\bar{\bar{\gamma}}$)) may be used in common to compute bounds on the optimal solution of (LN). As a matter of fact, they can be solved cyclically, using as input for one problem the output of the other, previously solved. Unfortunately, this procedure might not converge towards the optimal solution of (LN).

6. THE LINEAR RESTRICTIVE MODEL

Restrictive formulations correspond to variants of the problem where each client has to be assigned to one depot only. Note that, for the classical uncapacitated location problem (SPLP), one does not need to explicitly distinguish between the two formulations, since the (SPLP) model has the single assignment propriety.

When balancing requirements are included however, this propriety is not generally verified and the distinction has to be made. Moreover, several restrictive models are possible, depending upon the type of single assignment requirement : for all products or just a subset of them. We present hereafter two extreme restrictive formulations.

We assume again that supply, demand and costs are nonnegative and that the interdepot transportation costs verify the triangle inequality. We then formulate the models directly, without first stating and proving the propositions (corresponding to Proposition 1) which justify them.

The most restrictive formulation assigns each client to one depot exclusively, for all its transactions. Let :

$$x_{ij} = \begin{cases} 1 & \text{if client } i \text{ is assigned to depot } j \\ 0 & \text{otherwise} \end{cases} \qquad i \in C,\ j \in D$$

$$x = (x_{ij})$$

$$c_{ijp}(x) = c_{ijp} O_{ip} x_{ij}$$

$$c_{jip}(x) = c_{jip} D_{jp} x_{ij} \qquad i \in C,\ j \in D,\ p \in P$$

and the model is :

$$\text{(LU)} \quad \text{Minimize} \quad z = \sum_{j \in D} f_j y_j + \sum_{p \in P} \sum_{i \in C} \sum_{j \in D} (c_{ijp} O_{ip} + c_{jip} D_{ip}) x_{ij} + \sum_{p \in P} \sum_{j \in D} \sum_{k \in D} s_{jkp} w_{jkp}$$

subject to

$$\sum_{j\in D} x_{ij} = 1 \qquad \text{for all } i\in C$$

$$x_{ij} \leq y_j \qquad \text{for all } i\in C,\ j\in D$$

$$\sum_{i\in C} O_{ip}x_{ij} + \sum_{k\in D} w_{kjp} - \sum_{i\in C} D_{ip}x_{ij} - \sum_{k\in D} w_{jkp} = 0 \qquad \text{for all } j\in D,\ p\in P$$

$$w_{jkp} \geq 0 \qquad \text{for all } j,k\in D,\ p\in P$$

$$x_{ij}\in\{0,1\} \qquad \text{for all } i\in C,\ j\in D$$

$$y_j\in\{0,1\} \qquad \text{for all } j\in D.$$

On the other hand, the disaggregate restrictive model allows for a unique assignment, for each product, in each direction, of clients to depots. Let :

$$x_{ijp(jip)} = \begin{cases} 1 & \text{if client i ships (receives) all vehicles of type p at (from) depot j} \\ 0 & \text{otherwise} \end{cases} \qquad i\in C,\ j\in D,\ p\in P$$

$$x = (x_{ijp},\ x_{jip})$$

$$c_{ijp}(x) = c_{ijp}O_{ip}x_{ijp}$$

$$c_{jip}(x) = c_{jip}D_{ip}x_{jip}$$

and the model is :

(LUD) Minimize $z = \sum_{j\in D} f_j y_j$

$$+ \sum_{p\in P}\Big[\sum_{i\in C}\sum_{j\in D}(c_{ijp}O_{ip}x_{ijp} + c_{jip}D_{ip}x_{jip})$$

$$+ \sum_{j\in D}\sum_{k\in D} s_{jkp}w_{jkp}$$

subject to

$$\sum_{j\in D} x_{ijp} = 1$$

$$\sum_{j\in D} x_{jip} = 1 \qquad \text{for all } i\in C,\ p\in P$$

$$x_{ijp} \leq y_j \qquad \text{for all } i \in C,\ j \in D,\ p \in P$$

$$x_{jip} \leq y_j$$

$$\sum_{i \in C} O_{ip} x_{ijp} + \sum_{k \in D} w_{kjp} - \sum_{i \in C} D_{ip} x_{jip} - \sum_{k \in D} w_{jkp} = 0 \qquad \text{for all } j \in D,\ p \in P$$

$$w_{jkp} \geq 0 \qquad \text{for all } j,k \in D,\ p \in P$$

$$x_{ijp},\ x_{jip} \in \{0,1\} \qquad \text{for all } i \in C,\ j \in D,\ p \in P$$

$$y_j \in \{0,1\} \qquad \text{for all } j \in D.$$

It is unnecessary, at this time, to detail all intermediary formulations. Depending on the particular application, one or the other model will be appropriate. It is however quite clear that all these formulations share the same structure and that an efficient algorithm for one will solve all others as well. The computing effort could, however, be quite different, the number of integer variables growing with the degree of disaggregation of the model.

Note also that the restrictive formulations share the network structure previously observed in nonrestrictive models. It therefore would seem possible to extend to the restrictive models the characterization and algorithmic results attained for the general nonrestrictive case.

7. CONCLUSIONS

We have presented and analyzed the properties of a formulation of a multimode multicommodity location-distribution problem with inter-depot balancing requirements.

The model displays a number of theoretically and practically very interesting characteristics. A network structure emerges continually throughout the analysis, not only in the model itself, but also, and most

importantly, in all subproblems, relaxations, dualizations, etc. Given the richness and the well-known general efficiency of network models, this augurs well for the future development of efficient algorithms for our problem. We believe that it is more appropriate to develop specific algorithms for this problem than to use traditional algorithms for location models with general additional constraints. As for the algorithmic approach, several methods are possible and may advantageously build upon the structure and properties displayed by our model : enumeration, decomposition, dualization, relaxation, etc. We are presently exploring some of these avenues and others will be undertaken soon. Results will be communicated as soon as they become available.

Two extensions of the linear model present considerable interest and will be studied further : the integration of both loaded and empty movements into a single formulation and the consideration of capacities at all, or some, depots. And, of course, we also plan to look, in the near future, into the stochastic and nonlinear aspects of the problem.

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ANNEX

Here follows the proof of Proposition 1 of section 3, which states that when the interdepot transportation cost functions verify

$$s_{ikp} \leq s_{ijp} + s_{jkp} \qquad \text{for all } i,j,k \in D, p \in P$$

then constraints (3) of model (P1)

$$\left.\begin{aligned} w_{jkp} &\leq My_j \\ w_{kjp} &\leq My_j \end{aligned}\right\} \qquad \text{for all } j,k \in D, p \in P$$

are redundant.

Proof Call (P2) the model (P1) without constraints (3).

Let $v^* = (y^*, x^*, w^*)$ be an optimal solution to (P2).

We demonstrate that, either v^* is also optimal for (P1) or, one can deduce from v^* another solution $\bar{v}$ of equal objective value, that is optimal for (P1).

Two alternatives are possible :

i) v^* verifies the constraints (3) of (P1). It follows that v^* is optimal for (P1), (P2) being a relaxation of (P1).

ii) v^* does not satisfy these constraints.

Then, there are, at least, $r \in P$, $\ell, h \in D$ such that

$$w^*_{\ell hr} \neq 0 \quad \text{and} \quad y^*_\ell = 0.$$

But (by constraints (2))

$$y^*_\ell = 0 \Rightarrow x^*_{i\ell p} = x^*_{\ell i p} = 0 \qquad \text{for all } i \in C,\ p \in P$$

implying

$$\sum_{i \in C} x^*_{i\ell p} = \sum_{i \in C} x^*_{\ell i p} = 0 \qquad \text{for all } p \in P$$

therefore (by constraints (4))

$$\sum_{k \in D} w^*_{k\ell p} - \sum_{k \in D} w^*_{\ell k p} = 0 \qquad \text{for all } p \in P$$

and, in particular

$$\sum_{k \in D} w^*_{k\ell r} - \sum_{k \in D} w^*_{\ell k r} = 0 \quad ,$$

with

$$\sum_{k \in D} w^*_{k\ell r} \neq 0 \quad \text{and} \quad \sum_{k \in D} w^*_{\ell k r} \neq 0$$

since, by hypothesis,

$$w^*_{\ell h r} \neq 0 \quad \text{and} \quad w^*_{k\ell r} > 0,\ w^*_{\ell k r} > 0 \quad \text{for all } k \in D \ .$$

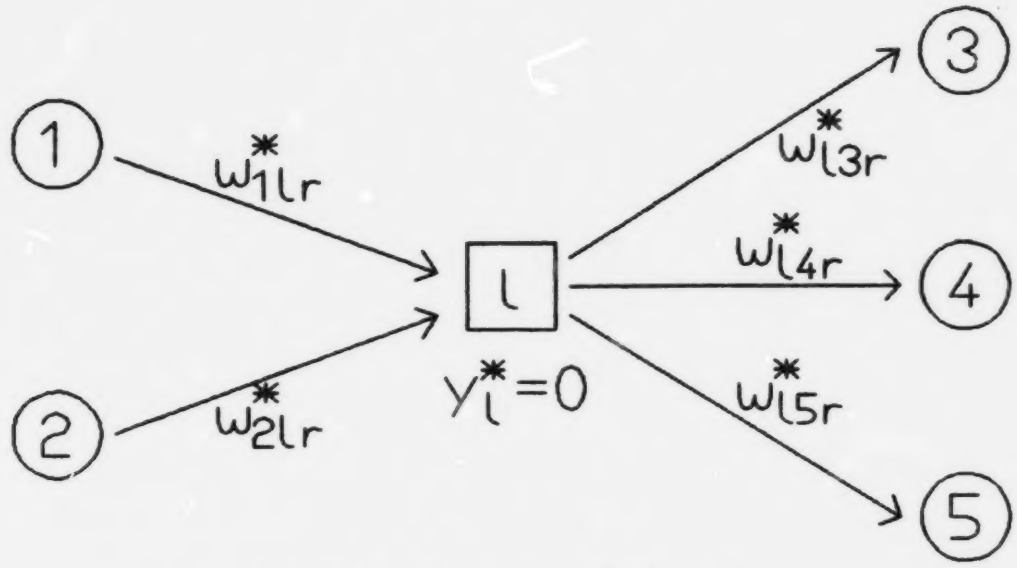
Define then the sets

$$D_I = \{k \mid w^*_{k\ell r} \neq 0\} \quad \text{et} \quad D_0 = \{k \mid w^*_{\ell k r} \neq 0\}$$

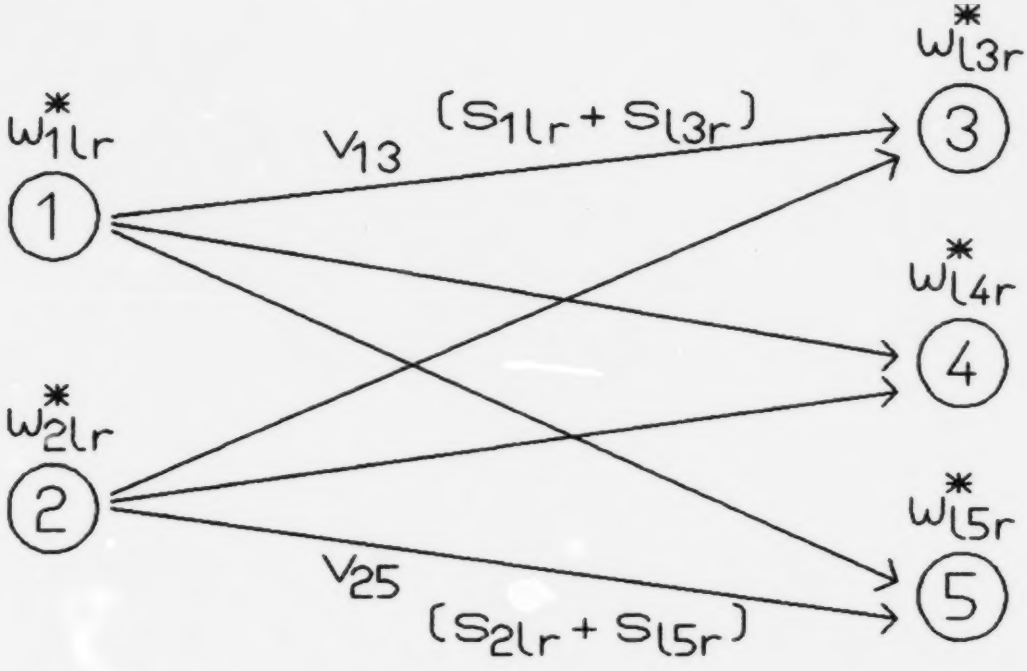
and build a bipartite transportation network $G = (N,A)$ (figure 2) where :

- $N = D_I \cup D_0$
- $A = D_I \times D_0$
- demand at nodes in D_0 = flows $w^*_{\ell k r}$
- supply at nodes in D_I = flows $w^*_{k\ell r}$
- costs on arcs $(k_1,k_2) \in A$ = $s_{k_1 \ell r} + s_{\ell k_2 r}$.

An optimal solution to the transportation problem, defined by the previous network, may be used to build a new solution $\bar{v} = (\bar{x},\bar{w})$ of (P1) with $\bar{x} = x^*$ and $\bar{w}$ such that :



a) initial network for solution v*



b) associated transportation network

Figure 2 : Example of a transportation network associated to a violated constraint of type (3)

$$\bar{w}_{kjp} = \begin{cases} w^*_{kjp} & \text{if } (k,j)\in D^*, \quad \text{for all } p \\ 0 & \text{if } p=r,\ j=\ell,\ k\in D_I \text{ or if } p=r,\ k=\ell,\ j\in D_0 \\ w^*_{kjp}+v^*_{kj} & \text{if } p=r,\ k\in D_I,\ j\in D_0 \end{cases}$$

where $D^* = D\times D \setminus \{(k,j) \mid k\in D_I \cup \{\ell\},\ j\in D_0 \cup \{\ell\}\}$.

We have therefore :

$$\sum_{p\in P}\sum_{k\in D}\sum_{j\in D} s_{kjp}\,\bar{w}_{kjp} = \sum_{p\in P}\sum_{(k,j)\in D^*} s_{kjp}\,w^*_{kjp} + \sum_{k\in D_I}\sum_{j\in D_0} s_{kjr}(w^*_{kjp}+v^*_{kj}) \ .$$

Yet, by hypothesis (6) :

$$\sum_{k\in D_I}\sum_{j\in D_0} s_{kjr}v^*_{kj} \le \sum_{k\in D_I}\sum_{j\in D_0}(s_{k\ell r}+s_{\ell jr})v^*_{kj}$$

$$= \Big(\sum_{k\in D_I} s_{k\ell r}\underbrace{\sum_{j\in D_0} v^*_{kj}}_{w^*_{k\ell r}}\Big) + \Big(\sum_{j\in D_0} s_{\ell jr}\underbrace{\sum_{k\in D_I} v^*_{kj}}_{w^*_{\ell jr}}\Big)$$

$$= \sum_{k\in D} s_{k\ell r}w^*_{k\ell r} + \sum_{j\in D} s_{\ell jr}w^*_{\ell jr}$$

Thus :

$$\sum_{p\in P}\sum_{k\in D}\sum_{j\in D} s_{kjp}\bar{w}_{kjp} \le \sum_{p\in P}\sum_{k\in D}\sum_{j\in D} s_{kjp}w^*_{kjp}$$

$$\Rightarrow\ z(\bar{v}) \le z(v^*)$$

In addition, since $\bar{v}$ is feasible for (P1), (P2) is a relaxation of (P1) and v^* is optimal for (P2), we have that

$$z(\bar{v}) = z(v^*).$$

□